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A CONSISTENT CRACKED BAR VIBRATION THEORY

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A consistent continuous cracked bar vibration theory is developed. The stress and displacement field about the crack was used to modify the stress and displacement field throughout the bar, and reduction to one spatial dimension was achieved by integrating the stress and displacement fields throughout the bar cross-sections so that the total displacement would be exact. The resulting linear differential equation with variable coefficients has the modified displacement field due to the crack imbedded in it. Any number of cracks can be introduced into the differential equation as modifications of the displacement field. A numerical solution and a first order perturbation solution are presented for the prediction of changes in longitudinal vibration natural frequencies of a fixed–free bar with a single open-edge transverse crack. To assess the validity of the assumptions made experiments on aluminum bars with fatigue cracks were performed. The analytical results correlate very closely with experimental results with better correlation than the local flexibility solution.

1. STATE OF THE ART

A crack on an elastic structural element introduces considerable local flexibility due to the strain energy concentration in the vicinity of the crack tip under load. This effect was recognized long ago, and the idea of an equivalent spring, a local compliance, was used to quantify, in a macroscopic way, the relation between the applied load and the strain concentration around the tip of the crack [1, 2]. This idea was mainly implemented in methods for experimentally determining a stress intensity factor, describing the intensity of the stress field about the tip of the crack, by measuring the local compliance of a cracked beam and relating it by energy arguments to the strain energy concentration. This became a standard method for experimental determination of the stress intensity factor, and a wealth of results—both analytical and experimental—were tabulated for a number of cases, different in loading and geometry [3].

The local flexibility, computed from known expressions for the stress intensity factor from fracture mechanics was introduced to the vibration analysis of rotors, cracked beams and plates by Dimarogonas [4, 5], Chondros [6] and Chondros and Dimarogonas [7, 8].

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The stress and displacement distributions at the crack tip have been extensively investigated in the past 40 years. The redistribution of stress in a body due to the presence of a crack may be studied by methods of linear elastic stress analysis. The high stress about the crack tip is usually accompanied by at least some plastic deformation and other non-linear effects. Linear elastic stress analysis properly forms the basis of most current fracture analysis for at least "small scale yielding" where all substantial non-linearity is confined within a linear elastic field surrounding the crack tip.

Barr [9] and Christides and Barr [10, 11] developed a cracked Euler–Bernoulli beam theory by deriving the differential equation and associated boundary conditions for a uniform Euler–Bernoulli beam containing one or more pairs of symmetric cracks. The reduction to one spatial dimension was achieved by using integrations over the cross-section after certain stress, strain, displacement and momentum fields were chosen. In particular, the modification of the stress field induced by the crack was introduced through a local experimental function which assumed an exponential decay with the distance from the crack and included a parameter that had to be evaluated by experiments.

It is possible to compute the displacement field about the crack by fracture mechanics methods, thus computing the disturbance of the field and developing a mathematical model for longitudinal vibration of a continuous cracked bar without the need of specific experimental data. This is subject of this investigation.

2. THE CONTINUOUS CRACKED BAR

A bar with a single transverse surface crack is shown in Figure 1, here with fixed–free boundary conditions. Motion along the length of the bar only will be considered. Moreover, the crack will be assumed always open. Let the displacement components be denoted by u_i, the strain components by γ_{ij} and the stress components by σ_{ij} with $i, j = 1, 2, 3$ referring to Cartesian axes x, y and z. Let p_i be the momentum such that $T_m = \frac{1}{2}(\rho\delta_{ij}p_i p_j)$ will be the kinetic energy density (δ_{ij} is Kronecker's delta). For arbitrary independent variations δu_i, $\delta\gamma_{ij}$, $\delta\sigma_{ij}$ and δp_i, the extended Hu–Washizu variational principle [9, 12, 13] is introduced in the form

$$\int_V \{[\sigma_{ij,j} + F_i - \rho\dot{p}_i]\delta u_i + [\sigma_{ij} - W_{,\gamma_{ij}}]\delta\gamma_{ij} + [\gamma_{ij} - (1 - \tfrac{1}{2}\delta_{ij})(u_{i,j} + u_{j,i})]\delta\sigma_{ij}$$

$$+ [\rho\dot{u}_i - T_{m,p_i}]\delta p_i\}\mathrm{d}V + \int_{S_g} [\bar{g}_i - g_i]\delta u_i \,\mathrm{d}S + \int_{S_u} [u_i - \bar{u}_i]\delta g_i \,\mathrm{d}S = 0, \qquad (1)$$

where $W(\gamma_{ij})$ is the strain energy density function, ρ is the density of the material. F_i, g_i and u_i are, respectively, the body forces, the surface traction and the surface displacement. Moreover, V is the total volume of the solid, S_g and S_u are its external surfaces, $T(x, t)$

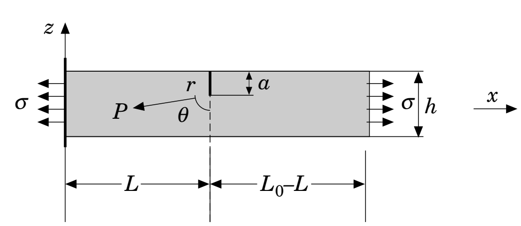

Figure 1. The geometry of the fixed–free bar with an edge crack.

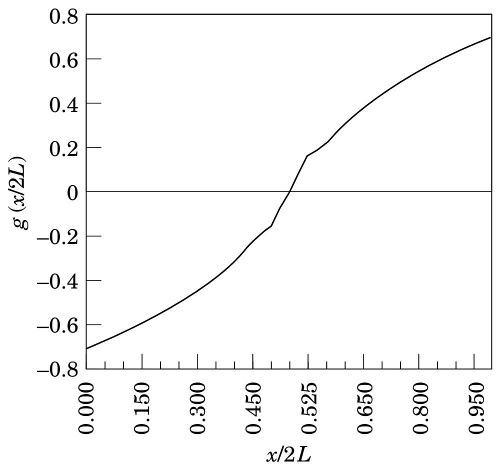

Figure 2. The crack disturbance function $g(x/2L) = 2f(x)(L_0)^{1/2}/C_0C(\alpha)$ from $x = 0$ to $x = L_0 = 2L$.

is the stress function, and $S(x, t)$ is the strain function. The overbar denotes the prescribed values of the surface traction and the surface displacement. The prescribed surface tractions g_i are applied over the surface S_g and the prescribed displacements u_i are over S_u. Together, S_g and S_u make up the total surface of the solid. The differentiation with respect to time $(\partial/\partial t)$ is indicated by a dot. Commas in the subscripts indicate differentiation.

Since a one-dimensional bar vibration analysis is pursued here, the displacement field in the absence of crack is assumed in the form $u_x = u_0(x, t)$, $u_y = 0$, $u_z = 0$, where u_0 is the elongation of the bar in the absence of the crack. The strain field is assumed to have the form $\gamma_{xx} = S(x, t)$, $\gamma_{yy} = \gamma_{zz} = -v\gamma_{xx}$, $\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$, where v is the Poisson ratio. The assumptions for γ_{yy} and γ_{zz} allow anticlastic curvature to be developed freely. The stress field is taken to be such that the only normal stress along the bar axis is of the form $\sigma_{xx} = T(x, t)$. All other stresses are set to zero. Finally, the momentum or velocity field is assumed to have the form $p_x = P(x, t)$ and $p_y = p_z = 0$.

The change in stress, strain and displacement distributions due to the crack will be expressed by two crack disturbance functions, one for the shear stress $m(x)$, introduced by Christides and Barr [10], and one for the axial displacement disturbance function $f(x)$, introduced here. The stress disturbance function $m(x)$ applies to the normal stress σ_{xx}; the remaining normal stresses and the shear stresses are still taken to be zero. It is further assumed that the presence of the crack will alter the strain γ_{xx} by the same function $m(x)$ because of the one-dimensional continuum assumption. The displacement disturbance function $f(x)$ applies to the displacement u_x. Finally, for a uniform bar with a crack, the above assumptions are expressed in the forms

$$u_x = [1 + f(x)]u(x, t), \quad u_y = 0, \quad u_z = 0, \qquad p_x = P(x, t), \quad p_y = 0, \quad p_z = 0,$$

$$\gamma_{xx} = [1 + m(x)]S(x, t), \qquad \gamma_{yy} = \gamma_{zz} = -v\gamma_{xx}, \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0,$$

$$\sigma_{xx} = [1 + m(x)]T(x, t), \qquad \sigma_{xz} = \sigma_{xy} = \sigma_{zz} = \sigma_{yy} = \sigma_{yz} = 0, \quad F_x = F_y = F_z = 0. \quad (2)$$

Following the method introduced in reference [10], the assumptions of equations (2) will be substituted into the general variational theorem, equation (1), neglecting the body forces and performing the integrations over the assumed independent variations of the unknown functions u, P, S and T.

Thus, for an arbitrary and independent variation δT, the strain–displacement term in equation (1) becomes

$$\int_V \left[\gamma_{xx} - \frac{\partial u_x}{\partial x}\right]\delta\sigma_{xx}\,\mathrm{d}V = \int_V \frac{\mathrm{d}}{\mathrm{d}x}\,\{(1+m)S - [(1+f)u]'\}(1+m)A\,\delta T\,\mathrm{d}x \tag{3}$$

where A is the cross-sectional area of the bar.

The stress–strain term of equation (1) is

$$\int_V \left\{\left[\sigma_{xx} - \frac{\partial W}{\partial \gamma_{xx}}\right]\delta\gamma_{xx} - \frac{\partial W}{\partial \gamma_{yy}}\,\delta\gamma_{yy} - \frac{\partial W}{\partial \gamma_{zz}}\,\partial\gamma_{zz}\right\}\delta S\,\mathrm{d}V \tag{4}$$

where

$$W = \tfrac{1}{2}\lambda e^2 + G(\gamma_{xx}^2 + \gamma_{yy}^2 + \gamma_{zz}^2) + \tfrac{1}{2}G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2),$$

is the strain energy density, $e = \gamma_{xx} + \gamma_{yy} + \gamma_{zz}$ is the dilatation, $G = E/(2(1+v))$ is the shear modulus, E is Young's modulus, and $\lambda = vE/[(1+v)(1-2v)]$ is the Lamé constant.

Upon substituting in the stress–strain term (4) the various quantities from equation (2), this term simplifies to

$$\int_x (T - ES)(1+m)^2 A\,\delta S\,\mathrm{d}x \tag{5}$$

and the velocity momentum term can be written as

$$\int_V \left[\rho\,\frac{\partial u_x}{\partial t} - \frac{\partial T_m}{\partial p_x}\right]\delta p_x\,\mathrm{d}V = \int_x \frac{\mathrm{d}}{\mathrm{d}x}\,\{[(1+f)\dot{u} - P]\}\rho A\delta p\,\mathrm{d}x. \tag{6}$$

The first term in equation (1) is the dynamic equilibrium term, which will lead to the equation of motion.

$$\int_V \left[\frac{\partial \sigma_{xx}}{\partial x} - \rho\dot{p}_x\right]\delta u_x\,\mathrm{d}V = \int_x \left\{\frac{\mathrm{d}}{\mathrm{d}x}\,[(1+m)T] - \rho\dot{P}\right\}(1+f)A\delta u\,\mathrm{d}x, \tag{7}$$

upon using equations (2).

In the surface integral over S_g and S_u in equation (1), the lateral surfaces of the bar can be assumed to be free from external traction. All prescribed tractions on the lateral surfaces are zero. The surface force is obtained from the stress components as $g_i = \sigma_{ij}n_j$, where n_j is the direction cosine of the external outward unit normal to the surfaces with the co-ordinate directions. If the bar is uniform, the normal to its lateral surfaces will be at right angles to its axis, so that n_x is zero.

For the prescribed forces, the surface integral of equation (1) over the ends of the bar ($x = 0$ and $x = L_0$) takes the form

$$\left[\int_A (\overline{X}_1 - \sigma_{xx})\delta u\,\mathrm{d}A\right]_{x=L_0} - \left[\int_A (\overline{X}_2 + \sigma_{xx})\delta u\,\mathrm{d}A\right]_{x=0}, \tag{8}$$

and for the prescribed displacements, it is

$$\left[\iint_A (u-\bar{u}_1)\delta\sigma_{xx}\,\mathrm{d}A\right]_{x=L_0} - \left[\iint_A (u-\bar{u}_2)\delta\sigma_{xx}\,\mathrm{d}A\right]_{x=0}. \tag{9}$$

The entire variational statement for the longitudinal vibration of the cracked bar can now be assembled by using equations (1)–(9).

The variations δu, δP, δS and δT are regarded as independent. Therefore, equation (1) implies that for arbitrary values of these variations, each term multiplied by them in the volume integral must independently be zero. Thus, from equation (3),

$$S = \frac{1+f}{1+m}u' + \frac{f'}{1+m}u, \tag{10}$$

from the stress–strain term (5),

$$T = ES, \tag{11}$$

from equation (6),

$$P = (1+f)\dot{u}, \tag{12}$$

and, from equation (7),

$$[(1+m)T]' - \rho\dot{P} = 0. \tag{13}$$

Equations (11)–(13) yield

$$(E/\rho)\,(\partial^2 u/\partial x^2 + a_1\,\partial u/\partial x + a_2 u) - \partial^2 u/\partial t^2 = 0 \tag{14}$$

where

$$a_1 = \frac{2f'}{(1+f)}, \qquad a_2 = \frac{f''}{(1+f)}.$$

Finally, the governing equation for the longitudinal vibration of the cracked bar is

$$(E/\rho)\,\partial^2[(1+f)u]/\partial x^2 = \partial^2[(1+f)u]/\partial t^2 \tag{15}$$

Equation (15) demonstrates that the crack displacement disturbance function $f(x)$ is directly affecting the displacement u. The stress and strain disturbance function $m(x)$ is cancelled and does not appear in equation (15).

The associated boundary conditions appropriate to the equation of motion (15) are

$$\sigma_{xx}|_{x=L_0} = \bar{X}_2, \qquad u|_{x=L_0} = \bar{u}_2, \qquad \sigma_{xx}|_{x=0} = -\bar{X}_1, \qquad u|_{x=0} = 0. \tag{16}$$

3. THE CRACK DISTURBANCE FUNCTION

A fixed–free bar with an open-edge surface crack is shown in Figure 1. Since the width of the cross-section of the bar is of the same order of magnitude as the height, the deformation of the bar will be described as plane strain. A crack of depth a is located at $x = L$ from the left end.

The axial displacement υ at point P in the crack tip region is [1, 3]:

$$\upsilon(r,\theta) = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2 - 2\nu - \cos^2\frac{\theta}{2}\right), \tag{17}$$

where r and θ are the polar co-ordinates of a coordinate system with the origin at the crack tip (see Figure 1).

The stress intensity factor K_I for a beam with an edge crack under uniform tension σ (see Figure 1) is [3]

$$K_I = \sigma\sqrt{\pi a}F_I(\alpha), \qquad F_I(\alpha) = 1{\cdot}12 - 0{\cdot}231\alpha + 10{\cdot}55\alpha^2 - 21{\cdot}72\alpha^3 + 30{\cdot}39\alpha^4, \qquad (18)$$

which has an accuracy of $\pm 0{\cdot}5\%$ for any $\alpha \leqslant 0{\cdot}6$, where $\alpha = a/h$, a is the crack depth and h is the height of the cross-section of the beam.

For a cracked bar, the modified stress and displacement fields at a point located at x, due to the presence of the crack, is assumed in the form

$$u^*(x, t) = u_0(x, t)[1 + f(x)], \qquad \sigma^*(x, t) = \sigma(x, t)[1 + m(x)], \qquad (19a,b)$$

where $u_0(x, t)$ and $\sigma(x, t)$ are the displacement and axial stress of the rod if the crack were not present, and $f(x)$ and $m(x)$ are the displacement and stress disturbance functions due to the presence of the crack. Since they are assumed to be independent of time, these functions will be determined from the known distribution of stresses and strains in a cracked bar due to a constant load in the form of a uniform stress field σ. Moreover, since a one-dimensional cracked bar vibration theory is pursued here, the field of stresses and strains will be averaged across the height of the bar. The displacement disturbance will be found by appropriate averaging of the displacement function found by the fracture mechanics method.

Let the axial displacement due to the crack, parallel to the x-axis be $\upsilon(r, \theta)$. Since the displacement $\upsilon(r, \theta)$ is a function of r and θ, to obtain a uniform displacement across the height of the bar, the displacement $C_0\upsilon(|x - L|, \pi/2)$ will be used, where the scale factor C_0 accounts for the error introduced in the averaging of the displacements across the height of the bar (thus yielding a one-dimensional mathematical model) and will be computed later so that the total displacement will be exact, and $|x - L|$ is the distance from crack location. In this way, one superposes an averaged local two-dimensional displacement field due to the crack on the one-dimensional bar displacement of the classical theory for the longitudinal vibration of prismatic bars. Due to the constant stress σ along the bar for static end loading, the displacement of the uncracked bar will be $u_0(x, t) = \sigma|L - x|/E$, in respect to the location of the crack at $x = L$. Thus, the crack disturbance function $f(x)$ will be in accordance with the above reasoning and equation (4), set in the form of the additional displacement due to the crack,

$$f(x) = C_0\upsilon(|x - L|, \pi/2)/u_0(x, t), \qquad (20)$$

where $\upsilon(|x - L|, \pi/2)$ is the displacement field along the bar at the crack tip ($\theta = \pi/2$) (see equation (17)) and will be symmetric in respect with the distance $|x - L|$ from the crack tip.

4. NATURAL FREQUENCIES; EDGE CRACK—NUMERICAL SOLUTION

A cracked uniform bar as shown in Figure 1 is fixed at $x = 0$ and free at $x = L_0$. A transverse surface crack is located at $x = L$. The axial displacement is $u(x, t)$.

The boundary conditions are

$$u|_{x=0} = 0, \qquad \partial u/\partial x|_{x=L_0} = 0. \qquad (21)$$

The solution of equation (15) for the longitudinal vibration of the undamaged bar, $f(x) = 0$, is

$$\omega_n = \{(2n - 1)\pi/2L_0\}\sqrt{E/\rho}, \qquad n = 1, 2, 3, \ldots . \qquad (22)$$

Equation (15) will be solved for the longitudinal vibration of a bar with an open crack in accordance with the above boundary conditions. The equation for the vibration modes is

$$[(1+f)U]'' + (\omega_n^*/c)^2[(1+f)U] = 0, \tag{23}$$

where ω_n^* are the natural frequencies of the cracked bar, and $c^2 = E/\rho$ is a material constant. When the crack is not present, the crack disturbance function $f(x)$ will vanish, and the natural frequencies ω_n^* of the cracked bar will change to the natural frequencies ω_n of the uncracked bar; see equation (22).

The solution of equation (23) is

$$U(x) = \phi(x)\{G_n \cos(\omega_n^* x/c) + H_n \sin(\omega_n^* x/c)\}. \tag{24}$$

Here G_n and H_n are constants, and $\phi(x) = 1/[1+f(x)]$ is the mode disturbance function. Application of the boundary conditions yields the natural frequency equation

$$\phi'(L_0)\sin(\omega_n^* L_0/c) + \phi(L_0)(\omega_n^*/c)\cos(\omega_n^* L_0/c) = 0, \qquad n = 1, 2, \ldots, \infty. \tag{25}$$

To obtain solutions for the natural frequencies of the cracked bar from the implicit natural frequency equation (25), a Newton method was applied, the results of which will be presented later. If a crack is not present, $f(x)$ will vanish, the function $\phi(x)$ will reduce to unity, and the natural frequency equation (25) for cracked bars will reduce to the natural frequency equation for uncracked bars, equation (22).

5. NATURAL FREQUENCIES; EDGE CRACK—PERTURBATION SOLUTION

An alternative perturbation method is utilized to calculate approximately the solution of the implicit frequency equation (25). To this end, a frequency disturbance factor ε_n is introduced:

$$\omega_n^* = \omega_n(1-\varepsilon_n). \tag{26}$$

From equation (22) one has

$$\sin(L_0\omega_n/c) = \pm 1 \qquad \text{and} \qquad \cos(L_0\omega_n c) = 0.$$

Consequently, the following relations hold:

$$\cos(L_0\omega_n^*/c) = \cos\{(L_0\omega_n/c) - (L_0\varepsilon_n\omega_n/c)\} = \pm\sin(L_0\varepsilon_n\omega_n/c),$$

$$\sin(L_0\omega_n^*/c) = \sin\{(L_0\omega_n/c) - (L_0\varepsilon_n\omega_n/c)\} = \pm\cos(L_0\varepsilon_n\omega_n/c).$$

Thus, the natural frequency equation (25) changes to

$$\phi'(L_0)\cos\left(\frac{L_0\varepsilon_n\omega_n}{c}\right) + \frac{\phi(L_0)}{L_0}\left(\frac{L_0\omega_n}{c} - \frac{L_0\varepsilon_n\omega_n}{c}\right)\sin\left(\frac{L_0\varepsilon_n\omega_n}{c}\right) = 0, \qquad n = 1, 2, \ldots, \infty. \tag{27}$$

Equation (27) yields

$$(1-\varepsilon_n)\tan(G_n\varepsilon_n) = -B/G_n, \tag{28}$$

where $G_n = L_0\omega_n/c$ and $B = L_0\phi'(L_0)/\phi(L_0)$, or

$$B = -L_0F'(L_0)/[1+f(L_0)]. \tag{29}$$

By using a Taylor series expansion, keeping terms up to the second order and ignoring the higher order terms $O(\varepsilon_n^3)$, from equation (27) an approximate equation can be obtained:

$$L_0 \frac{\phi'}{\phi}\left[1-\tfrac{1}{2}\left(\frac{L_0\varepsilon_n\omega_n}{c}\right)^2\right]+\left(\frac{L_0\omega_n}{c}-\frac{L_0\varepsilon_n\omega_n}{c}\right)\frac{L_0\varepsilon_n\omega_n}{c}=0. \tag{30}$$

The solution of this second degree natural frequency equation for ε_n is

$$\varepsilon_n = \frac{1-\sqrt{1+8(2+B)B}/[(2n-1)\pi]^2}{2+B}, \tag{31}$$

in explicit form. The frequency disturbance factor ε_n is determined from equation (31), and the natural frequencies of the cracked bar from equation (26).

Equation (31) can be further simplified by assuming that, in the region of an open crack, it is expected to have $f(L_0) \ll 1$ and thus equation (29) reduces to

$$B = -L_0 f'(L_0), \tag{32}$$

where the crack disturbance function derivative $f'(L_0)$ at $x = L_0$ can be derived from equation (20).

Since a perturbation solution is sought here, the constants C_0 and $C(\alpha)$ can be calculated by keeping the first term of the stress intensity factor in equation (18), thus yieldding

$$C_0 = 2\pi h(1-v^2)1{\cdot}12^2\alpha^2/C(\alpha)[\sqrt{L}+\sqrt{L_0-L}] \tag{33}$$

and

$$C(\alpha) = \sqrt{a}1{\cdot}12(\tfrac{3}{2}-2v)(1+v). \tag{34}$$

From equation (32) for the crack disturbance function derivative at $x = L_0$, one has

$$B = -1{\cdot}12^2\alpha^2\pi h(1-v^2)\sqrt{L_0-L}/[\sqrt{L}+\sqrt{L_0-L}]L_0, \tag{35}$$

where $\alpha = a/h$ is the ratio of the crack depth to the cross-section height.

Now, equation (31) can be further simplified by ignoring some less signficant terms for small crack depths. Since $B \ll 2$, as shown in equation (35), equation (31) can be reduced to

$$\varepsilon_n = -4B/[(2n-1)\pi]^2. \tag{36}$$

The frequency shifting ratio can now be calculated from equation (36), as a function of the crack depth and the ratio of the cross-section height to length. The frequency shifting for longitudinal vibration of a cracked bar is proportional to the square of the ratio of the crack depth to the cross-section height ($\sim\alpha^2$); see equations (26), (35) and (36).

6. BAR WITH LUMPED CRACK FLEXIBILITY

The above procedure distributes the added flexibility due to the crack over the length of the bar. For comparison, the natural frequencies of a cracked bar with the crack considered as a lumped local flexibility (see equation (20)) will now be computed.

Upon assuming that the crack flexibility is lumped at the location of the crack, the bar can be treated as two uniform bars, connected by a linear spring of local flexibility a_c at the crack location [15]. The modes of harmonic vibration on the two segments of the bar, left and right of the crack respectively, are

$$U_1(x) = A_1 \cos(\lambda x) + A_2 \cosh(\lambda x) + A_3 \sin(\lambda x) + A_4 \sinh(\lambda x),$$
$$U_2(x) = B_1 \cos(\lambda x) + B_2 \cosh(\lambda x) + B_3 \sin(\lambda x) + B_4 \sinh(\lambda x), \quad (37)$$

where the origin of x for both segments is at the left support, $\lambda = \omega_{Ln}/c$, $c^2 = E/\rho$, and ω_{Ln} are the natural frequencies of the cracked bar with lumped crack flexibility.

The coefficients A_i and B_i can be found by substituting this solution in the boundary conditions equations. The boundary conditions for the left and right parts of the beam are

$$u_1|_{x=0} = 0, \quad u_1|_{x=L} = 0, \quad \partial u_2/\partial x|_{x=L_0} = 0, \quad \partial u_1/\partial x|_{x=L} = \partial u_2/\partial x|_{x=L},$$
$$\partial u_2/\partial x|_{x=L} - \partial u_1/\partial x_{x=L} = (EAa_c)L_0 \partial u_2/\partial x|_{x=L}, \quad (38)$$

where EAa_c is the non-dimensional cracked section flexibility. From the above boundary conditions the natural frequency equation for the bar with lumped crack flexibility is found to be

$$\cos(\lambda L_0) + \lambda L_0 EAa_c \cos(\lambda L)[\sin(\lambda L)\cos(\lambda L_0) - \cos(\lambda L)\sin(\lambda L_0)] = 0. \quad (39)$$

Equation (39) can be solved numerically by Newton iteration to yield the cracked bar natural frequencies ω_{Ln}, as shown in Figure 3.

7. EXPERIMENTAL EVIDENCE

Prismatic aluminum bars were prepared for experimentation with length $L_0 = 0{\cdot}220$ m, height $h = 0{\cdot}0254$ m and thickness $b = 0{\cdot}006$ m. Each bar was fixed at one end on a heavy steel plate of a shaker table and at the other end a small accelerometer (1 g) was placed.

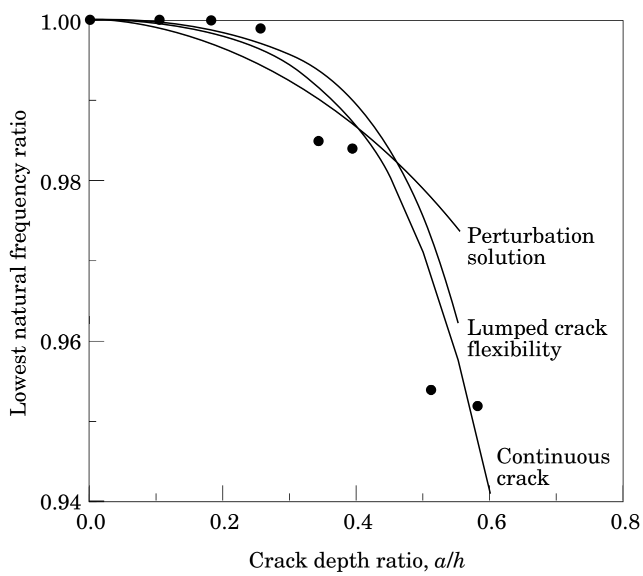

Figure 3. The lowest longitudinal natural frequency ratio ω/ω_0 for a fixed–free aluminum prismatic bar (with length $L_0 = 0{\cdot}220$ m and cross-section height $h = 0{\cdot}0254$ m) with an edge crack at mid-span. Numerical results are presented for the continuous cracked bar model equation (25), the perturbation solution equation (26) and the lumped crack flexibility model equation (39), and also experimental results (●).

At mid-span ($L = 0{\cdot}110$ m) a small notch was introduced to serve as a crack initiation point, and the bar was subsequently vibrated at its fundamental lateral natural frequency to force a crack formation and propagation. Specimens were prepared in this way with relative crack depths up to 60%. Then, the shaker was driven by a variable frequency power supply to locate the natural frequency by observing the measured longitudinal vibration amplitude. The natural frequency was recorded versus the crack depth.

8. RESULTS FOR A FIXED–FREE BAR WITH AN EDGE CRACK

The continuous cracked bar model equation (25), the perturbation solution equation (37), the local crack flexibility model equation (39), and experimental results from a cracked aluminum prismatic bar with an edge crack are plotted in Figure 3. As expected, the natural frequency changes in both models are greater for large crack depth. However, for relatively small crack depths ($a/h < 0{\cdot}35$), which are of importance in most crack identification problems, the perturbation solution appears to be adequate. For larger crack depths the continuous cracked bar model provides better accuracy than the lumped crack flexibility model.

9. CONCLUSIONS

In the continuous cracked beam theory developed by Christides and Barr [10, 11], a very important step towards a consistent cracked beam theory, an empirically defined crack disturbance function is used. In the continuous cracked beam theory developed by Wauer [16], a rigorous formalization of the local flexibility approach, the normalization of the local flexibility is used to develop a differential equation for the cracked beam. In the present formulation, no *a priori* assumptions are made for the stress field and the differential equation for longitudinal vibration of the cracked bar is based on analytical solutions for the stress field obtained by well established methods in fracture mechanics and published experimental or analytical results for the stress intensity factor.

The continuous cracked bar vibration theory developed here has led to a better approximation for the longitudinal vibration of cracked bars. Alternatively, a perturbation approach provides results of acceptable accurary for small crack depths. The experimental results are closer to the continuous crack formulation.

Since the stress and displacement fields in a cracked bar is two-dimensional and a one-dimensional theory is developed here, an averaging approximation had to be used. The error in this approximation and in the first order expansion of the stress field near the crack tip was controlled by requiring that the total displacement of one end of the bar in respect to the other is correct.

It is expected that the continuous cracked bar theory will be a useful alternative tool for vibration analysis of cracked structures, as it can easily be extended to other vibration modes, geometries and boundary conditions and to lateral and torsional vibration problems. The same general approach is used as in the Christides and Barr continuous beam lateral or torsional vibration theory, but no *a priori* assumptions are made for the stress field and well known results from fracture mechanics are used without the need to resort to specific experimentation.

The perturbation solution compares well with the continuous crack model and the lumped crack flexibility model solutions for crack depths up to 40%, which is adequate for practical considerations. For deeper cracks, the perturbation method underestimates the natural frequency changes, as expected. The local flexibility approach is also shown to underestimate the change in natural frequency for medium crack depths. The method developed here leads directly to a new differential equation and boundary conditions for a continuous cracked bar. Moreover, the differential equation formulation lends itself to further analysis, beyond the natural frequency calculation.

Finally, the continuous cracked bar formulation can be readily extended to multiple cracks and other geometries and boundary conditions.

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